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Linear Differential Equations

1. Solve $\frac{d^4y}{dx^4} - y = x\sin x.$

Soln: For CF

$$\left(\frac{d^4}{dx^4} - 1\right)y = 0$$

$$\Rightarrow D^4 - 1 = 0$$

$$\Rightarrow (D^2+1)(D^2-1) = 0 \Rightarrow (D+1)(D-1)(D+i)(D-i) = 0$$

$$\Rightarrow D = i, -i, 1, -1$$

$$\therefore CF = c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{x} + c_4 e^{-x}$$

$$= A\cos x + B\sin x + c_3 e^{x} + c_4 e^{-x} \quad (1)$$

$$\left[\because c_1 e^{ix} + c_2 e^{-ix} = c_1(\cos x + i\sin x) + c_2(\cos x - i\sin x)\right.$$

$$= \underbrace{(c_1+c_2)}_{A}\cos x + i\sin x(c_1 - c_2)$$

Let $A = c_1 + c_2$ and $B = i(c_1 - c_2)$

$$\left.\therefore \quad = A\cos x + B\sin x.\right]$$

For PI

$$PI = \frac{1}{(D^4-1)}x\sin x$$

$\therefore PI = \frac{1}{(D^2+1)(D^2-1)} x \sin x$

$\boxed{\text{Important step}}$

$\Rightarrow PI = \text{Imaginary part of } \frac{1}{(D^4-1)} x(\cos x + i \sin x)$

$= \text{I.P. of } \frac{1}{(D^4-1)} e^{ix} . x$

$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^4 - 1} x$

[D के स्थान पर D+i ($= D+a$ यहाँ $e^{ix} = e^{ax}$ $\therefore a = i$)]

$\Rightarrow PI = \text{I.P. of } e^{ix} . \frac{1}{D^4 + 4D^3 i + 6D^2 i^2 + 4D i^3 + i^4 - 1} x$

[Binomial theorem से expand करेंगे।

$(a+x)^4 = a^4 + {}^4C_1 a^3 x + {}^4C_2 a^2 x^2 + {}^4C_3 a x^3 + x^4$

$\therefore PI = \text{I.P. of } e^{ix} \frac{1}{D^4 + 4D^3 i - 6D^2 - 4Di + 1 - 1} x$

$\Rightarrow PI = \text{I.P. of } e^{ix} . \frac{1}{-4Di\left[1 - \left(\frac{6D^2 - 4D^3 i - D^4}{4Di}\right)\right]} x$

$$\Rightarrow PI = I.P.\ of\ e^{ix} \frac{1}{-4Di}\left[1-\left(-\frac{3D}{2}i - D^2 + \frac{1}{4}D^3 i\right)\right]^{-1} x$$

$$PI = I.P\ of\ e^{ix} \cdot \frac{i}{4D}\left[1+\left(\frac{3Di}{2} + D^2 - \frac{1}{4}D^3 i\right)\right]^{-1} x$$

$$\Rightarrow PI = I.P.\ of\ e^{ix} \cdot \frac{i}{4D}\left[1-\left(\frac{3Di}{2} + D^2 - \frac{1}{4}D^3 i\right) + \text{higher powers of } D\right] x$$

$$= I.P.\ of\ \frac{ie^{ix}}{4D}\left[x - \frac{3i}{2}\right]$$

$$= I.P.\ of\ \frac{ie^{ix}}{4}\int\left(x - \frac{3}{2}i\right) dx$$

$$= I.P\ of\ \frac{ie^{ix}}{4}\left[x^2 - \frac{3x}{2}\right]$$

$$= I.P\ of\ \frac{1}{4}\left[i\cos x - \sin x\right]\left(x^2 - \frac{3x}{2}\right)$$

$$\Rightarrow PI = \frac{1}{4}\cos x\left(x^2 - \frac{3x}{2}\right) \quad \text{--- } (2)$$

$\therefore$ complete solution is given by

$$y = CF + PI$$

where CF is given by eq (1)

and PI is given by eq (2).